COMPLEX LIE ALGEBRAS CORRESPONDING TO WEIGHTED PROJECTIVE LINES

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Abstract. The aim of this paper is to give an alternative proof of Kac's theorem for weighted projective lines ([5]) over the complex field. The geometric realization of complex Lie algebras arising from derived categories ([8]) is essentially used.

1. Introduction

It is well known that the dimension vectors of indecomposable representations of quiver Q correspond $1-1$ to the positive roots of the Kac-Moody algebra associated to Q.

In [5], Crawley-Boevey proved an analogue of Kac's Theorem as follows:

Theorem 1.1. *If $\mathbb{X}_{\mathbf{p},\underline{\lambda}}$ is a weighted projective line over an algebraically closed field K and $\alpha \in \hat{Q}$, there is an indecomposable sheaf in Coh($\mathbb{X}_{\mathbf{p},\underline{\lambda}}$) of type α if and only if α is a positive root. Moreover, there is a unique indecomposable for a real root, infinitely many for an imaginary root.*

This theorem describes the possible dimension vectors of indecomposable sheaves. In order to prove it, Crawley-Boevey reduced to the case when K is the algebraic closure of a finite field. He worked over a finite field F_q and associated a Lie algebra L to the category of coherent sheaves on a weighted projective line over this finite field. We note that the Lie algebra L is defined over a field F, which has characteristic l such that $q = 1$ in F.

We find that the proof can be simplified when K is changed to the complex field $\mathbb{C}$. Using [8] and the derived equivalence between the category of coherent sheaves on a weighted projective line and the module category of the corresponding canonical algebra, we construct a Lie algebra L on the category of coherent sheaves on a weighted projective line over $\mathbb{C}$ and find elements which satisfy the relations of the loop algebra. We calculate the Euler characteristics instead of counting numbers.

Let v be a vertex of the star-shaped graph (see 3.2) and write α_v for the simple root corresponding to v. Let $e \in L_{\alpha_v}$, $f \in L_{-\alpha_v}$, using the standard arguments in Lie algebra over the base field $\mathbb{C}$, we have the isomorphism $L_\phi \simeq L_{s_v(\phi)}$, i.e, the simple reflection induces isomorphism. Finally, we reduce to three simple cases by a sequence of reflections which were solved in [6].

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We note that in the process of the proof of the Kac Theorem on weighted projective lines, the operator $\theta = exp(\mathrm{ad}\ e)exp(-\mathrm{ad}\ f)exp(\mathrm{ad}\ e)$ in the sl_2-representation can be defined directly and the definition trouble occurring in the case of the finite field is avoided. Moreover, the process of finding a suitable field as the base field of the Lie algebra can be omitted. This simplifies the proof.

2. Lie algebras arising from derived categories

2.1. Let Λ be a finite dimensional and finite global dimensional associative algebra over $\mathbb{C}$. We can write (up to Morita equivalence) $\Lambda = \mathbb{C}Q/J$, where Q is a quiver and J is the admissible ideal generated by a set R of relations.

Consider the category modΛ of finite dimensional Λ-modules and its bounded derived category $D^b(\Lambda)$. In [8], Xiao, Xu and Zhang obtained a geometric realization of complex Lie algebras arising from the root category $D^b(\Lambda)/(T^2)$. We will give a short review here.

2.2. We fix $\{P_1, P_2, \cdots, P_l\}$ to be a complete set of indecomposable projective Λ-modules. A complex $C^\bullet$ of Λ-modules is called a period-2 complex if it satisfies $C^\bullet[2] = C^\bullet$. Let $P^\bullet = (P^0, P^1, \partial_0, \partial_1)$ be a period-2 complex of projective Λ-modules such that each P^i has the decomposition $P^i = \bigoplus_{j=1}^{l} e_j^i P_j$. We denote by $\underline{e}(P^i)$ the vector $(e_1^i, e_2^i, \cdots, e_l^i)$, then $\underline{\mathbf{e}} = (\underline{e}(P^0), \underline{e}(P^1))$ is called the projective dimension sequence of $P^\bullet$. We define $\mathcal{P}_2(\Lambda, \underline{\mathbf{e}})$ to be the subset of

$$\mathrm{Hom}_\Lambda(P^0, P^1) \times \mathrm{Hom}_\Lambda(P^1, P^0)$$

which consists of (∂_0, ∂_1) such that $\partial_0\partial_1 = 0$ and $\partial_1\partial_0 = 0$.

The algebraic group $G_{\underline{\mathbf{e}}} = \mathrm{Aut}_\Lambda(P^0) \times \mathrm{Aut}_\Lambda(P^1)$ acts on $\mathcal{P}_2(\Lambda, \underline{\mathbf{e}})$ by conjugation action. Thus two projective complexes in $\mathcal{P}_2(\Lambda, \underline{\mathbf{e}})$ are in the same orbit under the $G_{\underline{\mathbf{e}}}$-action if and only if they are quasi-isomorphic.

Let K_0 be the Grothendieck group of $D^b(\Lambda)$, also of $D^b(\Lambda)/(T^2)$. There is a canonical surjection from the abelian group of projective dimension sequences to K_0, which will be denoted by $\underline{\dim}$. We define $\mathcal{P}_2(\Lambda, \mathbf{d}) = \bigcup_{\underline{\mathbf{e}} \in \underline{\dim}^{-1}(\mathbf{d})} \mathcal{P}_2(\Lambda, \underline{\mathbf{e}})$ for any $\mathbf{d} \in K_0$. Then $\mathcal{P}_2(\Lambda, \mathbf{d})$ has a natural topological structure induced by that of $\mathcal{P}_2(\Lambda, \underline{\mathbf{e}})$, see [8] for details. Thus $G_{\mathbf{d}} = \bigcup_{\underline{\mathbf{e}} \in \underline{\dim}^{-1}(\mathbf{d})} G_{\underline{\mathbf{e}}}$ partially acts on $\mathcal{P}_2(\Lambda, \mathbf{d})$. Moreover, we set

$$T_{\underline{\mathbf{e}}} = \{t_x^{\pm} | x \in \mathcal{P}_2(\Lambda, \underline{\mathbf{e}}) \text{ is constructible}\}$$

and $T = \bigcup_{\underline{\mathbf{e}} \in \underline{\dim}^{-1}(\mathbf{0})} T_{\underline{\mathbf{e}}}$ whose action on $\mathcal{P}_2(\Lambda, \mathbf{d})$ is also partially defined. With the groupoid $\langle G_{\mathbf{d}}, T\rangle$ acting on $\mathcal{P}_2(\Lambda, \mathbf{d})$, we have that

$$\mathcal{QP}_2(\Lambda, \mathbf{d}) = \mathcal{P}_2(\Lambda, \mathbf{d})/\sim = \mathcal{P}_2(\Lambda, \mathbf{d})/\langle G_{\mathbf{d}}, T\rangle$$

where $x \sim y$ in $\mathcal{P}_2(\Lambda, \mathbf{d})$ if and only if their corresponding complexes are quasi-isomorphic.

2.3. We denote by $M(X)$ the set of all constructible functions on an algebraic variety X with values in $\mathbb{C}$. The set $M(X)$ is naturally a $\mathbb{C}$-linear space. Let G be an algebraic group acting on X. Then we denote by $M_G(X)$ the subspace of $M(X)$ consisting of all G-invariant functions.

Let $\mathbf{d}$ be a dimension vector in K_0 and $\mathcal{O}$ be a $\langle G_{\mathbf{d}}, T\rangle$-invariant and support-bounded constructible subset of $\mathcal{P}_2(\Lambda, \mathbf{d})$. Here support-bounded means there exists a projective dimension sequence $\underline{\mathbf{e}}$ such that $\mathcal{O} = \langle G_{\mathbf{d}}, T\rangle(\mathcal{O} \cap \mathcal{P}_2(\Lambda, \underline{\mathbf{e}}))$ and $\underline{\mathbf{e}}$ is called a support projective dimension sequence of $\mathcal{O}$.

We define the function $1_{\mathcal{O}} : \mathcal{P}_2(\Lambda, \mathbf{d}) \to \mathbb{C}$ given by taking values 1 on each point in $\mathcal{O}$ and 0 otherwise. A function f on $\mathcal{P}_2(\Lambda, \mathbf{d})$ is called $\langle G_{\mathbf{d}}, T\rangle$-invariant constructible function if f can be written as a sum of finite sums $\sum_i m_i 1_{\mathcal{O}_i}$ where $m_i \in \mathbb{C}$ and any $\mathcal{O}_i$ is $\langle G_{\mathbf{d}}, T\rangle$-invariant and support-bounded constructible subset of $\mathcal{P}_2(\Lambda, \mathbf{d})$. Let $\underline{\mathbf{e}}_1$ and $\underline{\mathbf{e}}_2$ be projective dimension sequences in $\underline{\dim}^{-1}(\mathbf{d})$. Two constructible functions $f_i \in M_{G_{\underline{\mathbf{e}}_i}}(\mathcal{P}_2(\Lambda, \underline{\mathbf{e}}_i))$, $i = 1, 2$ are equivalent if there exists a $\langle G_{\mathbf{d}}, T\rangle$-invariant constructible F over $\mathcal{P}_2(\Lambda, \mathbf{d})$ such that $f_i = F|_{\mathcal{P}_2(\Lambda, \underline{\mathbf{e}}_i)}$, $i = 1, 2$. Let $f \in M_{G_{\underline{\mathbf{e}}}}(\Lambda, \underline{\mathbf{e}})$ and $\underline{\mathbf{e}} \in \underline{\dim}^{-1}(\mathbf{d})$. The equivalent class of f is denoted by $\hat{f}$. Let $M_{GT}(\mathcal{P}_2(\Lambda, \mathbf{d}))$ be the space of the equivalence classes $\hat{f}$ of constructible functions f over $\mathcal{P}_2(\Lambda, \underline{\mathbf{e}})$ for any $\underline{\mathbf{e}} \in \underline{\dim}^{-1}(\mathbf{d})$.

An equivalence class $\hat{f} \in M_{GT}(\mathcal{P}_2(\Lambda, \mathbf{d}))$ is called indecomposable if any point in $supp(f)$ is indecomposable in the (relative) homotopy category of all period-2 complexes of projective modules. Let $I_{GT}(\mathbf{d})$ be the $\mathbb{C}$-space of all indecomposable equivalence classes in $M_{GT}(\mathcal{P}_2(\Lambda, \mathbf{d}))$.

Let $\mathcal{O}_1 \subset \mathcal{P}_2(\Lambda, \underline{\mathbf{e}}'') \subset \mathcal{P}_2(\Lambda, \mathbf{d}_1)$ and $\mathcal{O}_2 \subset \mathcal{P}_2(\Lambda, \underline{\mathbf{e}}') \subset \mathcal{P}_2(\Lambda, \mathbf{d}_2)$ be $G_{\underline{\mathbf{e}}''}$- and $G_{\underline{\mathbf{e}}'}$-invariant constructible set, respectively. For $L \in \mathcal{P}_2(\Lambda, \underline{\mathbf{e}}' + \underline{\mathbf{e}}'')$, we set

$$W(\mathcal{O}_1, \mathcal{O}_2; L) = \{(f, g, h) | Y \xrightarrow{f} L \xrightarrow{g} X \xrightarrow{h} Y[1] \text{ is a distinguished triangle}$$

$$\text{with } X \in \mathcal{O}_1, Y \in \mathcal{O}_2\},$$

then the quotient space $W(\mathcal{O}_1, \mathcal{O}_2; L)/G_{\underline{\mathbf{e}}''} \times G_{\underline{\mathbf{e}}'}$ is independent of choices of support projective dimension sequences of both $\langle G_{\mathbf{d}_1}, T\rangle \mathcal{O}_1$ and $\langle G_{\mathbf{d}_2}, T\rangle \mathcal{O}_2$. So we denote it by $V(\mathcal{O}_1, \mathcal{O}_2; L)$.

Thus the convolution multiplication $\hat{1}_{\mathcal{O}_1} * \hat{1}_{\mathcal{O}_2} \in M_{GT}(\mathcal{P}_2(\Lambda, \mathbf{d}_1 + \mathbf{d}_2))$ can be defined as follows:

$$\hat{1}_{\mathcal{O}_1} * \hat{1}_{\mathcal{O}_2}(L) = F^L_{\mathcal{O}_1 \mathcal{O}_2} := \chi(V(\mathcal{O}_1, \mathcal{O}_2; L))$$

where χ denotes the quasi Euler characteristic of quotient space as in [8].

We set $\mathfrak{n} = \bigoplus_{d \in K_0} I_{GT}(\mathbf{d})$ and $\mathfrak{h} = K_0 \otimes_{\mathbb{Z}} \mathbb{C}$ which is spanned by $\{h_{\mathbf{d}} | \mathbf{d} \in K_0\}$. The symmetric Euler bilinear form on $\mathfrak{h}$ is given as

$$(h_{\mathbf{d}_1} | h_{\mathbf{d}_2}) = \dim_{\mathbb{C}} \mathrm{Hom}(X, Y) - \dim_{\mathbb{C}} \mathrm{Hom}(X, Y[1])$$

$$+ \dim_{\mathbb{C}} \mathrm{Hom}(Y, X) - \dim_{\mathbb{C}} \mathrm{Hom}(Y, X[1])$$

for any $X \in \mathcal{P}_2(\Lambda, \mathbf{d_1})$, $Y \in \mathcal{P}_2(\Lambda, \mathbf{d_2})$.

Thus $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}$ becomes a Lie algebra over $\mathbb{C}$ with the Lie bracket $[-, -]$ defined below.

$$[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}] = [\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]_{\mathfrak{n}} + \chi(\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]}) h_{\mathbf{d}_1}$$

where $\overline{\mathcal{O}_1 \cap \mathcal{O}_2[1]} \backsimeq (\mathcal{O}_1 \cap \mathcal{O}_2[1])_{\underline{\mathbf{e}}}/G_{\underline{\mathbf{e}}}$ for a support projective dimension sequence of $\mathcal{O}_1 \cap \mathcal{O}_2[1]$.

$$[\hat{1}_{\mathcal{O}_1}, \hat{1}_{\mathcal{O}_2}]_{\mathfrak{n}}(L) := F^L_{\mathcal{O}_1 \mathcal{O}_2} - F^L_{\mathcal{O}_2 \mathcal{O}_1}$$

$$[h_{\mathbf{d}_1}, \hat{1}_{\mathcal{O}_2}] := (h_{\mathbf{d}_1} | h_{\mathbf{d}_2}) \hat{1}_{\mathcal{O}_2}, \ [\hat{1}_{\mathcal{O}_2}, h_{\mathbf{d}_1}] := -(h_{\mathbf{d}_1} | h_{\mathbf{d}_2}) \hat{1}_{\mathcal{O}_2}$$

$$[h_{\mathbf{d}_1}, h_{\mathbf{d}_2}] := 0.$$

3. The category of coherent sheaves on weighted projective lines

3.1. **Weighted projective lines.** Let $\mathbf{p} = (p_1, p_2, \cdots, p_n) \in (\mathbb{N}^*)^n$ and $\underline{\lambda} = \{\lambda_1, \cdots, \lambda_n\}$ be a collection of distinct closed points on the projective line $\mathbb{P}^1(\mathbb{C})$.

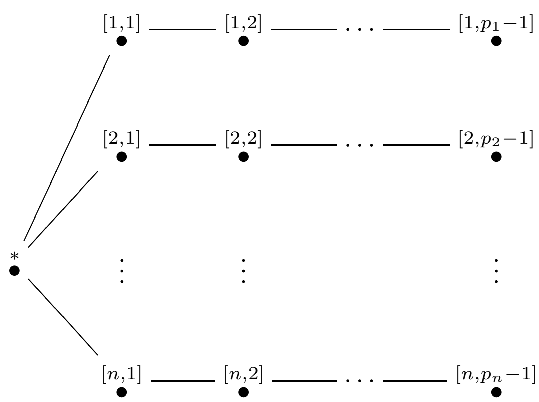
Instead of giving the definition, we give a description of the structure of the category $\mathrm{Coh}(\mathbb{X}_{\mathbf{p},\underline{\lambda}})$ (see [2] for details).

Let $\mathscr{F}$ and $\mathscr{T}$ be two full extension-closed subcategories of $\mathrm{Coh}(\mathbb{X}_{\mathbf{p},\underline{\lambda}})$. For any sheaf $\mathscr{M} \in \mathrm{Coh}(\mathbb{X}_{\mathbf{p},\underline{\lambda}})$, it can be decomposed as $\mathscr{M}_t \oplus \mathscr{M}_f$ where $\mathscr{M}_t \in \mathscr{T}$ and $\mathscr{M}_f \in \mathscr{F}$ and $\mathrm{Hom}(\mathscr{M}_t, \mathscr{M}_f) = \mathrm{Ext}^1(\mathscr{M}_f, \mathscr{M}_t) = 0$ for any $\mathscr{M}_t \in \mathscr{T}$ and $\mathscr{M}_f \in \mathscr{F}$.

The category $\mathscr{T}$ decomposes as a coproduct $\mathscr{T} = \coprod_{x \in \mathbb{X}_{\mathbf{p},\underline{\lambda}}} \mathscr{T}_x$, where $\mathscr{T}_x$ is equivalent to the category $\mathrm{rep}_0(C_{r_x})$ consisting of nilpotent representations of the cyclic quiver with r_x vertices, where $r_x = p_i$ if $x = \lambda_i$, $1 \le i \le n$, and $r_x = 1$ otherwise.

The category $\mathscr{F}$ has a filtration by objects of the form $\mathscr{O}(\vec{x})$, where $\vec{x} \in L(\mathbf{p}) = \mathbb{Z}\vec{x}_1 \oplus \mathbb{Z}\vec{x}_2 \oplus \cdots \oplus \mathbb{Z}\vec{x}_n / J$ where J is the submodule generated by $\{p_1\vec{x}_1 - p_s\vec{x}_s | s = 2, \cdots, n\}$. Set $\vec{c} = p_1\vec{x}_1 = \cdots = p_n\vec{x}_n \in L(\mathbf{p})$. For $\mathscr{O}(r\vec{c})$, there is a unique simple objects $S_{i,0}$ in each $\mathscr{T}_{\lambda_i}$ with $\mathrm{dimHom}(\mathscr{O}(r\vec{c}), S) = 1$. The simple objects are S_a $(a \in \mathbb{P}^1 \backslash \underline{\lambda})$ and $S_{i,j}$ $(1 \le i \le n, 0 \le j \le p_i - 1)$, which satisfy the relations $\mathrm{dimExt}(S_{i,j}, S_{i,j-1}) = 1$.

3.2. **Star-shaped graph and loop algebra.** Associating to the weight type $(\mathbf{p}, \underline{\lambda})$, we have a star-shaped graph Γ:



whose vertex set $\mathcal{I}$ consists of the central vertex $*$ and vertices in n branches which are denoted by $[i,j]$, $1 \le i \le n$, $1 \le j \le p_i - 1$.

Consider the Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(\Gamma)$ associated to the graph Γ. We have the *loop algebra* of $\mathfrak{g}$, denoted by $\mathcal{L}\mathfrak{g}$, which is defined to be the complex Lie algebra generated by $h_{i,k}, e_{i,k}, f_{i,k} : i \in \mathcal{I}, k \in \mathbb{Z}$ and c subject to the following relations:

$$
\begin{aligned}
&[h_{i,k}, h_{j,l}] = k\delta_{k,-l} a_{ij} c,\\
&[e_{i,k}, f_{j,l}] = \delta_{i,j} h_{i,k+l} + k\delta_{k,-l} c,\\
&[h_{i,k}, e_{j,l}] = a_{ij} e_{j,l+k},\ [h_{i,k}, f_{j,l}] = -a_{ij} f_{j,l+k},\\
&[e_{i,k}, e_{i,l}] = 0,\ [f_{i,k}, f_{i,l}] = 0,\quad c\ \ \textit{central}\\
&[e_{i,k_1}, [e_{i,k_2}, [\ldots, [e_{i,k_n}, e_{j,l}] \ldots] = 0,\ \text{for } n = 1 - a_{ij},\\
&[f_{i,k_1}, [f_{i,k_2}, [\ldots, [f_{i,k_n}, f_{j,l}] \ldots] = 0,\ \text{for } n = 1 - a_{ij}.
\end{aligned}
$$

The root systems of $\mathfrak{g}$ and $\mathcal{L}\mathfrak{g}$ are denoted by Δ and $\hat{\Delta}$ respectively and the root lattices are denoted by Q and $\hat{Q} = Q \oplus \mathbb{Z}\delta$. In view of the graph Γ, the simple roots in Δ are denoted by α_* and α_{ij} for $1 \leq i \leq n$ and $1 \leq j \leq p_i - 1$. We also know that $\hat{\Delta} = \mathbb{Z}^*\delta \cup (\Delta + \mathbb{Z}\delta)$.

There is a natural identification of $\mathbb{Z}$-modules $K_0(\mathrm{Coh}(\mathbb{X})) \cong \hat{Q}$ given by

$$[S_{i,j}] \mapsto \alpha_{ij}, \text{ for } j = 1, \cdots, p_i - 1, \quad [S_{i,0}] \mapsto \delta - \sum_{j=1}^{p_i-1} \alpha_{ij}, \quad [\mathscr{O}(k\vec{c})] \mapsto \alpha_* + k\delta.$$

Naturally, the non-negative combinations of the elements α_{ij}, $\delta - \sum_{j=1}^{p_i-1} \alpha_{ij}$, $\alpha_* + k\delta$ and δ form the positive cone $\hat{Q}_+$.

3.3. **Derived equivalence and the Lie algebra.** In [3], Ringel introduced the class of canonical algebras attached to $(\mathbf{p}, \underline{\lambda})$. It is well known that there is a triangle equivalence $D^b(\mathrm{Coh}(\mathbb{X}_{\mathbf{p},\underline{\lambda}})) \simeq D^b(\mathrm{mod}(\Lambda^{op}_{\mathbf{p},\underline{\lambda}}))$ where $\mathrm{Coh}(\mathbb{X}_{\mathbf{p},\underline{\lambda}})$ is a hereditary abelian category. Therefore, their root categories are equivalent. We simply write Λ for $\Lambda^{op}_{\mathbf{p},\underline{\lambda}}$. Then by 2.3, we can define a $\hat{Q}$-graded complex Lie algebra L on the root category of Λ.

The set of indecomposable objects of $\mathcal{R}_{\mathbf{p},\underline{\lambda}} = D^b(\mathrm{Coh}(\mathbb{X}_{\mathbf{p},\underline{\lambda}}))/(T^2)$ is $\mathrm{ind}\mathcal{R}_{\mathbf{p},\underline{\lambda}} = (\mathrm{indCoh}(\mathbb{X}_{\mathbf{p},\underline{\lambda}}) \bigcup \{TY | Y \in \mathrm{indCoh}(\mathbb{X}_{\mathbf{p},\underline{\lambda}})\}$. For any simple object S, $S[r]$ denotes the unique object $S[r]$ with length r and top S for $r > 0$, and denotes the unique object TY for $r < 0$, where Y is of length $-r$ with $\mathrm{Ext}^1(Y, S) \neq 0$. H_r is the set of $X \in \mathrm{ind}\mathcal{R}_{\mathbf{p},\underline{\lambda}}$ of type $r\delta$ and with $\mathrm{Hom}(X, S_{i,j}) = 0$ for all $1 \leq i \leq n$ and $1 \leq j \leq p_i - 1$.

Lemma 3.1. *(i) For any $X \in ind\mathcal{R}_{\mathbf{p},\underline{\lambda}}$, the image of X in the root category of the canonical algebra Λ is denoted by $F(X)$. Assume $F(X) \in \mathcal{P}_2(\Lambda, \underline{e})$, $\hat{1}_{G_{\underline{e}}F(X)}$ is the equivalence class of the characteristic function of the orbit $G_{\underline{e}}F(X)$. Then $\hat{1}_{G_{\underline{e}}F(X)} \in I_{GT}(\underline{dim}\ \underline{e})$*

(ii) The set $F(H_r) \subset \mathcal{P}_2(\Lambda, \underline{e(r)})$, and $\hat{1}_{G_{\underline{e(r)}}F(H_r)} \in I_{GT}(\underline{dim}\ \underline{e(r)})$. Moreover, $\chi(G_{\underline{e(r)}}F(H_r)/G_{\underline{e(r)}}) = 2$.

Proof. (i) is trivial because $F(X)$ is also indecomposable in the root category of Λ. (ii) The Serre subcategory generated by $\mathscr{O}(k\vec{c})$ for $k \in \mathbb{Z}$, $S_a[l] (a \in \mathbb{P}^1 \backslash \underline{\lambda}, l \geq 1)$ and $S_{i,0}[lp_i]$ $(1 \leq i \leq n, l \geq 1)$ is equivalent to the category $\mathrm{Coh}(\mathbb{P}^1)$. Therefore, it is enough to prove the non-weighted case. We have $D^b(\mathrm{Coh}(\mathbb{P}^1)) = D^b(\mathrm{rep}\overrightarrow{Q})$, where $\overrightarrow{Q}$ is the Kronecker quiver. There exists $\underline{e(r)}$ such that $F(H_r) \subset \mathcal{P}_2(\Lambda, \underline{e(r)})$. The results in the Kronecker quiver case imply $\hat{1}_{G_{\underline{e(r)}}F(H_r)} \in I_{GT}(\underline{\dim}\ \underline{e(r)})$ and $\chi(G_{\underline{e(r)}}F(H_r)/G_{\underline{e(r)}}) = 2$. □

4. NEW PROOF

4.1. **Main result.**

Theorem 4.1. *If $\mathbb{X}_{\mathbf{p},\underline{\lambda}}$ is a weighted projective line over the complex field $\mathbb{C}$ and $\alpha \in \hat{Q}$, there is an indecomposable sheaf in $Coh(\mathbb{X}_{\mathbf{p},\underline{\lambda}})$ of type α if and only if α is a positive root. Moreover, there is a unique indecomposable for a real root, infinitely many for an imaginary root.*

This theorem is proved in [5] over any algebraically closed field. In the case of the complex field $\mathbb{C}$, we find a new proof as follows, which also uses the Hall

algebras. We define a $\hat{Q}$-graded complex Lie algebra L on the root category $\mathcal{R}_{\mathbf{p},\underline{\lambda}}$ (section 3.3) and there is a subalgebra satisfying the relations of the loop algebra.

Set $l(r)=1, for \ \ r\geq 0$ and $l(r)=-1, for \ \ r<0$. For any $X\in \mathrm{ind}\mathcal{R}_{\mathbf{p},\underline{\lambda}}$, we write $\hat{1}_{(X)}=\hat{1}_{G_{\underline{e}}F(X)}$ and $\hat{1}_{(H_r)}=\hat{1}_{G_{\underline{e(r)}}F(H_r)}$ for short.

Theorem 4.2. *The following elements satisfy the relations in $\mathcal{L}\mathfrak{g}$.*

$$e_{v,r}=\begin{cases} l(r)\hat{1}_{(S_{i,j}[rp_i+1])} & v=[i,j]\\ l(r)\hat{1}_{(\mathscr{O}(r\vec{c}))} & v=*\end{cases}\qquad f_{v,r}=\begin{cases} l(r-1)\hat{1}_{(S_{i,j-1}[rp_i-1])} & v=[i,j]\\ l(r)\hat{1}_{(\mathscr{O}(-r\vec{c}))} & v=*\end{cases}$$

$$c=-\delta \quad h_{v,r}=\begin{cases} -\alpha_v & r=0\\ l(r)\hat{1}_{(S_{i,j}[rp_i])}-l(r)\hat{1}_{(S_{i,j-1}[rp_i])} & r\neq 0, v=[i,j]\\ l(r)\hat{1}_{(H_r)} & r\neq 0, v=*\end{cases}$$

4.2. **Proof of Theorem 4.2.** We note that $[\hat{1}_{\mathcal{O}_1},\hat{1}_{\mathcal{O}_2}](M)=0$ for M decomposable and the triangles $X\to Y\to Z\to$ with $X,Y,Z\in \mathrm{ind}\mathcal{R}_{\mathbf{p},\underline{\lambda}}$ are in 1-1 correspondence with short exact sequences in $\mathrm{Coh}(\mathbb{X}_{\mathbf{p},\underline{\lambda}})$. The section 3 of [5] is still true for the complex field. However, we calculate the Euler characteristics instead of counting numbers.

(i)

$$[l(r)\hat{1}_{(S_{i,j}[r])},l(s)\hat{1}_{(S_{i,k}[s])}]=\begin{cases}\delta_{j-r,k}l(r+s)\hat{1}_{(S_{i,j}[r+s])}-\delta_{j,k-s}l(r+s)\hat{1}_{(S_{i,k}[r+s])} & r+s\neq 0\\ -\delta_{j-r,k}[S_{i,j}[r]] & r+s=0\end{cases}$$

Proof of (i): In one tube, if $0\to X\to Y\to Z\to 0$ is a short exact sequence of indecomposable objects, then there is a unique short exact sequence with the same terms up to automorphisms of any two of X,Y,Z. Using the fact $\chi(\text{one point})=1$, we complete the proof.

Note that we can prove all relations in one tube by (i) now.

(ii) $[h_{*,r},h_{*,-r}]=[l(r)\hat{1}_{(H_r)},l(-r)\hat{1}_{(H_{-r})}]=-r\delta\chi(G_{\underline{e(r)}}H_r/G_{\underline{e(r)}})=-2r\delta=2rc$

(iii) For $[e_{*r},f_{*,s}]$,

if $r+s=0$, $[e_{*r},f_{*,-r}]=-[\hat{1}_{(\mathscr{O}(r\vec{c})},\hat{1}_{(\mathscr{O}(-r\vec{c})}]=-\chi((\mathscr{O}(r\vec{c}))[\mathscr{O}(r\vec{c})]=-[\mathscr{O}(r\vec{c})]=h_{*,0}+rc$

if $r+s\neq 0$, assume $r+s>0$, we get the short exact sequence $0\to\mathscr{O}(-(r+s)\overrightarrow{c})\to\mathscr{O}\to Y\to 0$ with $Y\in H_{r+s}$, $\dim\mathrm{Hom}(\mathscr{O},Y)=r+s$. The non-epimorphisms form a subspace of dimension $r+s-1$ and each short exact sequence is determined by an epimorphism up to an automorphism of $\mathscr{O}(-(r+s)\overrightarrow{c})$. $[e_{*r},f_{*,s}](Y)=\chi(\mathbb{C}^{r+s-1})=1$. That implies $[e_{*r},f_{*,s}]=h_{*,r+s}$.

(iv)We assume $r>0$, The support of the function $[h_{[i,1],r},e_{*,s}]$ is the orbit of $\mathscr{O}((r+s)\overrightarrow{c})$. For $X\in(\mathscr{O}((r+s)\overrightarrow{c}))$, $[h_{[i,1],r},e_{*,s}](X)=-\chi(\text{one point})=-1$, then $[h_{[i,1],r},e_{*,s}]=-e_{*,r+s}$.

(v)We assume $r>0$, The support of the function $[h_{*,r},e_{*,s}]$ is the orbit of $\mathscr{O}((r+s)\overrightarrow{c})$. For $X\in(\mathscr{O}((r+s)\overrightarrow{c}))$, $[h_{*,r},e_{*,s}](X)=\chi(\mathbb{P}^1)=2$, then $[h_{*,r},e_{*,s}]=2e_{*,r+s}$. ∎

4.3. **Proof of Theorem 4.1.** L is a $\hat{Q}$-graded complex Lie algebra with $L_0 = \hat{Q} \bigotimes_{\mathbb{Z}} \mathbb{C}$. For $\phi \in \hat{Q}_+$, if there is an indecomposable sheaf X in $\mathrm{Coh}(\mathbb{X}_{\mathbf{p},\underline{\lambda}})$ of type ϕ, then $\hat{1}_{(X)} \in L_\phi$ and $L_\phi \neq 0$. If there is no indecomposable sheaf of type ϕ, $L_\phi = 0$. The case of $-\phi \in \hat{Q}_+$ is similar.

For $\phi \in \hat{Q}_+$, we want to determine whether or not $L_\phi = 0$. We need the following two lemmas:

Lemma 4.3. *Let v be a vertex of the star-shaped graph. The operators ad $e_{v,0}$ and ad $f_{v,0}$ are locally nilpotent.*

Proof. For any $\psi \in \hat{Q}$ and $f \in L_\psi$, we need to show $(\mathrm{ad}\ e_{v,0})^n(f) = (\mathrm{ad}\ f_{v,0})^n(f) = 0$, for some n. It is enough to prove $(\mathrm{ad}\ \hat{1}_X)^n(\hat{1}_Y) = 0$ for any two indecomposable sheaves X, Y with $\mathrm{Ext}^1(X,X) = 0$:

If Z is in the support of $(\mathrm{ad}\ \hat{1}_X)(\hat{1}_Y)$, then Z is the middle term of a nonsplit exact sequence whose end terms are X and Y, so

$\mathrm{dimExt}^1(X,Z) + \mathrm{dimExt}^1(Z,X) < \mathrm{dimExt}^1(X,Y) + \mathrm{dimExt}^1(Y,X)$, thus

$(\mathrm{ad}\ \hat{1}_X)^n(\hat{1}_Y) = 0$ for $n > \mathrm{dimExt}^1(X,Y) + \mathrm{dimExt}^1(Y,X)$. □

Lemma 4.4. *Let v be a vertex of the star-shaped graph and write α_v for the simple root corresponding to v. For any $\phi \in \hat{Q}_+$, we have $L_\phi \simeq L_{s_v(\phi)}$.*

Proof. As proved in 4.2, $e_{v,0} \in L_{\alpha_v}$ and $f_{v,0} \in L_{-\alpha_v}$ satisfy $[e_{v,0}, f_{v,0}] = h_{v,0}$ and for $f \in L_\psi$, $(\mathrm{ad}\ h_{v,0})(f) = (\alpha_v, \psi)f$. From Lemma 4.3, ad $e_{v,0}$ and ad $f_{v,0}$ are locally nilpotent. So the operator $\theta = exp(\mathrm{ad}\ e_{v,0})exp(-\mathrm{ad}\ f_{v,0})exp(\mathrm{ad}\ e_{v,0})$ acts on $h_{v,0}$ as multiplication by -1. For $f \in L_\phi$, we have $\theta(f) = \Sigma_{r\in\mathbb{Z}} f'_r$ with $f'_r \in L_{\phi + r\alpha_v}$.

$$\Sigma_{r\in\mathbb{Z}}(\alpha_v, \phi)f'_r = \theta([h_{v,0}, f]) = [\theta(h_{v,0}), \theta(f)] = [-h_{v,0}, \theta(f)]$$
$$= [-h_{v,0}, \Sigma_{r\in\mathbb{Z}} f'_r] = -\Sigma_{r\in\mathbb{Z}}(\alpha_v, \phi + r\alpha_v)f'_r$$

Comparing the coefficients of the above equation, we get $\theta(f) = f'_r$ with $r = -(\alpha_v, \phi)$, which means $\theta(L_\phi) \subseteq L_{\phi - (\alpha_v,\phi)\alpha_v}$. Similarly $\theta^{-1}(L_{\phi-(\alpha_v,\phi)\alpha_v}) \subseteq L_\phi$. Thus the operator $\theta = exp(\mathrm{ad}\ e_{v,0})exp(-\mathrm{ad}\ f_{v,0})exp(\mathrm{ad}\ e_{v,0})$ induces an isomorphism $L_\phi \simeq L_{s_v(\phi)}$. □

For $\phi \in \hat{Q}$, we can reduce to the following three cases by a sequence of reflections:

$\pm\alpha_v + r\delta$;

$\alpha + r\delta$, with α in the fundamental region;

$\alpha + r\delta$, where α is not positive or negative, or has disconnected support.

For the first case: $\mathrm{dim}L_\phi = \mathrm{dim}L_{\pm\alpha_v + r\delta} = 1$, there is a unique indecomposable sheaf;

the second case: $\mathrm{dim}L_\phi = \mathrm{dim}L_{\alpha+r\delta} = \infty$, there are infinitely many indecomposable sheaves (see [6]);

the last case: $\mathrm{dim}L_\phi = \mathrm{dim}L_{\alpha+r\delta} = 0$, there is no indecomposable object.

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